1. After the realization of the shock, the HH's problem is:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

s.t.

$$p_0 c_0 \le M_{-1} + T_0 \quad (\mu_0)$$

$$p_t c_t \le M_{t-1}, \ \forall t \ge 1 \quad (\mu_t)$$

$$M_t + B_t \le M_{t-1} + T_t - p_t c_t + w_t l_t + R_{t-1} B_{t-1} + \Pi_t, \ \forall t \ge 1$$

$$M_0 + B_0 \le M_{-1} + T_0 - p_0 c_0 + w_0 l_0 \quad (\lambda_0) + \Pi_0$$

c_0: $u_c(0) = \mu_0 p_0 + \lambda_0 p_0$ $\qquad M_0$: $\lambda_0 = \mu_1 + \lambda_1$

c_t: $\beta^t u_c(t) = (\mu_t + \lambda_t) p_t, \ t \ge 1$ $\qquad M_t$: $\lambda_t = \mu_{t+1} + \lambda_{t+1}, \ t \ge 1$

l_0: $u_l(0) = -\lambda_0 w_0$ $\qquad B_0$: $\lambda_0 = R_0 \lambda_1$

l_t: $\beta^t u_l(t) = -\lambda_t w_t, \ t \ge 1$ $\qquad B_t$: $\lambda_t = R_t \lambda_{t+1}, \ t \ge 1$

From period 1, the economy is stationary

$$\Rightarrow \beta = \frac{\lambda_t}{\lambda_{t+1}} = \frac{1}{R_{t+1}} \Rightarrow R_{t+1} = \frac{1}{\beta}, \ \forall t \ge 1. \Rightarrow R_t = \frac{1}{\beta}, \ \forall t \ge 0$$

and $\beta \cdot \frac{u_c(1)}{u_c(0)} = \frac{\lambda_0}{\lambda_0 + \mu_0} \cdot \frac{p^*}{p_0}$, $\quad \frac{\beta^t u_l(t)}{u_l(0)} = \frac{\lambda_t}{\lambda_0} \cdot \frac{w^*}{w_0} = \beta^t \frac{w^*}{w_0}$

$$\Rightarrow \frac{u_l(t)}{u_l(0)} = \frac{w^*}{w_0}, \text{ if we assume } u = \log c_t + \psi \log(1 - l_t)$$

$$\Rightarrow \frac{1 - l_0}{1 - l^*} = \frac{w^*}{w_0} \Rightarrow (1 - l_0) w_0 = w^* (1 - l^*)$$

and $\frac{c_0}{1 - l_0} = \frac{1}{1 + \frac{\mu_0}{\lambda_0}} \cdot \frac{w_0}{p_0}$

$$\frac{\beta^t u_c(t)}{u_c(0)} = \beta^t \cdot \frac{c_0}{c^*} = \frac{\beta^{t-1} \lambda_0}{\mu_0 + \lambda_0} \cdot \frac{p^*}{p_0}$$

$$\Rightarrow \beta \frac{c_0}{c^*} = \frac{c_0}{1 - l_0} \cdot \frac{p_0}{w_0} \cdot \frac{p^*}{p_0} = \frac{c_0}{1 - l_0} \frac{p^*}{w_0}$$

$$\Rightarrow \frac{\beta}{c^*} = \frac{p^*}{(1 - l_0) w_0}$$

Assumption: $u(c,e)$ is homothetic in (c,e)

Taking $\{P_t, \omega_t\}$ as given, $\bar{c}_0 = \frac{M_{-1}+T_0}{M_{-1}} \underline{c}_0$, $\bar{e}_0 = \frac{M_{-1}+T_0}{M_{-1}} \underline{e}_0$

come the problem when there is a positive shock of $\underline{c}_0, \underline{e}_0$ come the problem when there is no positive shock.

$\Rightarrow \bar{e}_0 > \underline{e}_0$, $\bar{c}_0 > \underline{c}_0 \Leftrightarrow \bar{y}_0 > \underline{y}_0$

$$\text{since } \begin{cases} (1-\bar{e}_0)\bar{\omega}_0 = \bar{\omega}^*(1-\bar{e}^*) \\ (1-\underline{e}_0)\underline{\omega}_0 = \underline{\omega}^*(1-\underline{e}^*) \end{cases} \Rightarrow \bar{\omega}_0 > \underline{\omega}_0 .$$

For sticky producers, the price is determined at period -1, so the price $\bar{p}_0, \underline{p}_0$ are:

$$\bar{p}_0 = \left[\alpha \cdot \left(\frac{\bar{\omega}_0}{\theta}\right)^{\frac{\theta}{\theta-1}} + \int_0^1 p_i^{s\,\frac{\theta}{\theta-1}} di\right]^{\frac{\theta-1}{\theta}}$$

$$\underline{p}_0 = \left[\alpha \left(\frac{\underline{\omega}_0}{\theta}\right)^{\frac{\theta}{\theta-1}} + \int_\alpha^1 p_i^{s\,\frac{\theta}{\theta-1}} di\right]^{\frac{\theta-1}{\theta}}$$

$\Rightarrow$ when $\bar{\omega}_0 > \underline{\omega}_0 \Rightarrow \bar{p}_0 > \underline{p}_0$

and
$$\frac{\bar{p}_0}{\bar{\omega}_0} = \left[\alpha \cdot \theta^{\frac{\theta}{1-\theta}} + \int_\alpha^1 \left(\frac{p_i^s}{\bar{\omega}_0}\right)^{\frac{\theta}{\theta-1}} di\right]^{\frac{\theta-1}{\theta}}$$

$$\frac{\underline{p}_0}{\underline{\omega}_0} = \left[\alpha\, \theta^{\frac{\theta}{1-\theta}} + \int_\alpha^1 \left(\frac{p_i^s}{\underline{\omega}_0}\right)^{\frac{\theta}{\theta-1}} di\right]^{\frac{\theta-1}{\theta}}$$

$\Rightarrow$ when $\bar{\omega}_0 > \underline{\omega}_0 \Rightarrow \frac{\bar{\omega}_0}{\bar{p}_0} > \frac{\underline{\omega}_0}{\underline{p}_0}$.

$\Rightarrow \bar{y}_0 > \underline{y}_0$; $\bar{p}_0 > \underline{p}_0$; $\frac{\bar{\omega}_0}{\bar{p}_0} > \frac{\underline{\omega}_0}{\underline{p}_0}$.

HW6 Macro III

Problem 2:

$$C_t + K_{t+1} = f(K_t) + (1-\delta) K_t$$

$$U'(C_t) = \beta U'(C_{t+1}) \left(f'(K_{t+1}) + 1 - \delta \right)$$

Log - linearize:

$$C \hat{c}_t + K \hat{k}_{t+1} = f'(K) K \hat{k}_t + (1-\delta) K \hat{k}_t \qquad ①$$

$$U''(C) C \hat{c}_t = \beta U''(C) C \hat{c}_{t+1} (f'(K) + 1 - \delta) + \beta U'(C) f''(K) K \hat{k}_{t+1} \qquad ②$$

Steady state relationship:

$$C + K = f(K) + (1-\delta) K \Rightarrow C + \delta K = f(K) \qquad ③$$

$$U'(C) = \beta U'(C) (f'(K) + 1 - \delta) \Rightarrow \beta (f'(K) + (1-\delta)) = 1 \qquad ④$$

$$④ \Rightarrow f'(K) = \frac{1}{\beta} + \delta - 1 \qquad K = f'^{-1}\left(\frac{1}{\beta} + \delta - 1\right)$$

$$③ \Rightarrow C = f\left(f'^{-1}\left(\frac{1}{\beta} + \delta - 1\right)\right) - \delta f'^{-1}\left(\frac{1}{\beta} + \delta - 1\right)$$

Then $① \Rightarrow \hat{k}_{t+1} = \frac{1}{\beta} \hat{k}_t - \frac{C}{K} \hat{c}_t$

$$② \Rightarrow \hat{c}_{t+1} + \frac{\beta U'(C) f''(K) K}{U''(C) C} \hat{k}_{t+1} = \hat{c}_t$$

$$\begin{pmatrix} 1 & \frac{\beta U'(C) f''(K) K}{U''(C) C} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{C}{K} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\beta u'(c) f''(K) K}{u''(c) c} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\frac{c}{K} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{\beta u'(c) f''(K) K}{u''(c) c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{K} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{\beta u'(c) f''(K)}{u''(c)} & -\frac{u'(c) f''(K) K}{u''(c) c} \\ -\frac{c}{K} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

$$\triangleq A \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

$\det(A - \lambda I) = 0 \quad \Rightarrow$

$$\left(1 + \frac{\beta u'(c) f''(K)}{u''(c)} - \lambda\right)\left(\frac{1}{\beta} - \lambda\right) - \frac{u'(c) f''(K)}{u''(c)} = 0$$

$$\Rightarrow \quad \lambda^2 - \left(\frac{1}{\beta} + \beta g + 1\right)\lambda + \frac{1}{\beta} = 0,$$

where $g = \dfrac{u'(c) f''(K)}{u''(c)} > 0$.

$$\left.\begin{array}{l} \lambda_1 + \lambda_2 = \frac{1}{\beta} + \beta g + 1 > 2 \\ \lambda_1 \cdot \lambda_2 = \frac{1}{\beta} > 1 \\ (\lambda_1 - 1)(\lambda_2 - 1) = \lambda_1\lambda_2 + 1 - (\lambda_1 + \lambda_2) = -\beta g < 0 \end{array}\right\} \begin{array}{l} \lambda_1, \lambda_2 \\ \Rightarrow \text{One} > 1 \\ 0 < \text{the other} \end{array}$$

The root $\in (0, 1)$ is converging; the other root > 1 is NOT.

. HH's problem:

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)\, u(c_t(s^t), l_t(s^t)).$$

s.t. $p_t(s^t)\, c_t(s^t) \le M_t(s^t)$ $\quad (\lambda_t(s^t))$

$$M_t(s^t) + B_t(s^t) \le M_{t-1}(s^{t-1}) - p_{t-1}(s^{t-1})\, c_{t-1}(s^{t-1})$$
$$+ R_{t-1}(s^{t-1})\, B_{t-1}(s^{t-1}) + \pi_t(s^t)$$
$$+ w_t(s^t)\, l_t(s^t) + T_t. \quad (\mu_t(s^t))$$

F.O.C: $c_t(s^t)$: $\beta^t \pi_t(s^t)\, u_c(c_t(s^t), l_t(s^t)) = p_t(s^t)\left(\lambda_t(s^t) + \sum_{s^{t+1}} \mu_{t+1}\right.$

$l_t(s^t)$: $\beta^t \pi_t(s^t)\, u_l(c_t(s^t), l_t(s^t)) = -\sum_{s^{t+1}} \mu_{t+1}(s^{t+1})\, w_t(s^t)$

$M_t(s^t)$: $\mu_t(s^t) = \lambda_t(s^t) + \sum_{s^{t+1}} \mu_{t+1}(s^{t+1})$.

$B_t(s^t)$: $\mu_t(s^t) = R_t(s^t) \sum_{s^{t+1}} \mu_{t+1}(s^{t+1})$.

$$\Rightarrow -\frac{u_l(s^t)}{u_c(s^t)} = \frac{w_t(s^t)}{p_t(s^t)} \frac{\sum_{s^{t+1}} \mu_t(s^{t+1})}{\lambda_t(s^t) + \sum_{s^{t+1}} \mu_{t+1}(s^{t+1})} = \frac{w_t(s^t)}{p_t(s^t)} \frac{1}{R_t(s^t)} \quad ①$$

$$\frac{\beta^{t+1} \pi_{t+1}(s^{t+1})\, u_c(s^{t+1})}{\beta^t \pi_t(s^t)\, u_c(s^t)} = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} \cdot \frac{\mu_{t+1}(s^{t+1})}{\mu_t(s^t)}.$$

$$\Rightarrow \beta\, \pi_{t+1}(s^{t+1}|s^t) \frac{u_c(s^{t+1})}{u_c(s^t)} = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} \cdot \frac{\mu_{t+1}(s^{t+1})}{\mu_t(s^t)}$$

$$\Rightarrow \sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\, \beta \frac{u_c(s^{t+1})}{u_c(s^t)} \cdot \frac{p_t(s^t)}{p_{t+1}(s^{t+1})} = \frac{1}{R_t(s^t)}.$$

$$\Rightarrow \sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\, \beta \frac{u_c(s^{t+1})}{u_c(s^t)} \frac{p_t(s^t) R_t(s^t)}{p_{t+1}(s^{t+1})} = 1. \quad ②$$

. ~~intermediate~~ goods producer:
Final.

$$\max_{y^f(i,s^t),\, y^s(i,s^t)} p_t(s^t)\, y_t(s^t) - \left[\int_0^{\alpha} p^f(i, s^t)\, y^f(i, s^t)\, di + \int_{\alpha}^{1} p^s(i|s^{t-1})\, y^s(i, \ldots) di \right]$$

s.t. $y_t(s^t) = \left(\int_0^{\alpha} y^f(i, s^t)^{\theta} di + \int_{\alpha}^{1} y^s(i, s^t)^{\theta} di \right)^{\frac{1}{\theta}}.$

$$\Rightarrow P_t(s^t)\frac{1}{\theta}\, y(s^t)^{1-\theta}\cdot \theta\, y^f(i,s^t)^{\theta-1} = p^f(i,s^t).$$

$$\Rightarrow y^f(i,s^t) = \left[\frac{p^f(i,s^t)}{P_t(s^t)}\right]^{\frac{1}{\theta-1}} \cdot y_t(s^t)$$

Similarly, $y^s(i,s^t) = \left(\frac{p^s(i,s^{t-1})}{P_t(s^t)}\right)^{\frac{1}{\theta-1}} y_t(s^t).$

Flexible intermediate goods producer:

$$\max_{p^f(i,s^t)} \left(p^f(i,s^t) - w_t(s^t)\right) p^f(i,s^t)^{\frac{1}{\theta-1}}\, y_t(s^t)\, P_t(s^t)^{\frac{1}{1-\theta}}.$$

$$\Rightarrow p^f(i,s^t) = \frac{w_t(s^t)}{\theta}.$$

Sticky intermediate goods producer:

$$\max_{p^s(i,s^t)} \sum_{s^{t+1}} Q_{t+1}(s^{t+1}|s^t)\left(p^s(i,s^t) - w_{t+1}(s^{t+1})\right) p^s(i,s^t)^{\frac{1}{\theta-1}} \frac{y_{t+1}}{P_{t+1}(s^{t+1})}$$

$$\Rightarrow p^s(i,s^t) = \frac{1}{\theta}\, \frac{\sum_{s^{t+1}} Q_{t+1}(s^{t+1}|s^t)\, P_{t+1}(s^{t+1})^{\frac{1}{1-\theta}}\, w_{t+1}(s^{t+1})\, y_{t+1}(s^{t+1})}{\sum_{s^{t+1}} Q_{t+1}(s^{t+1}|s^t)\, P_{t+1}(s^{t+1})^{\frac{1}{1-\theta}}\, y_{t+1}(s^{t+1})}.$$

and $Q_{t+1}(s^{t+1}|s^t) = \beta\, \pi_{t+1}(s^{t+1}|s^t)\, \frac{u_c(s^{t+1})}{u_c(s^t)}\, \frac{P_t(s^t)}{P_{t+1}(s^{t+1})}.$

- Market clearing: goods: $c_t(s^t) = y_t(s^t)$.

  labor: $l_t(s^t) = \left[\alpha\left[\frac{p^f(i,s^t)}{P_t(s^t)}\right]^{\frac{1}{\theta-1}} + (1-\alpha)\left(\frac{p^s(i,s^{t-1})}{P_t(s^t)}\right)^{\frac{1}{\theta-1}}\right.$

  $\cdot\, y_t($

log linearize the system:

First: $p_t(s^t)^{\frac{\theta}{\theta-1}} = \alpha\, p_f(i, s^t)^{\frac{\theta}{\theta-1}} + (1-\alpha)\, p_s(i, s^{t-1})^{\frac{\theta}{\theta-1}}$

log-linearizing this, we have:

$$p_t^{*\frac{\theta}{\theta-1}}(1+\lambda p_t) = \alpha\, p_{ft}^{*\frac{\theta}{\theta-1}}(1+\lambda_1 p_{ft}) + (1-\alpha)\, p_{st}^{*\frac{\theta}{\theta-1}}(1+\lambda_2 p_{st})$$

In steady state, we have $p_t^* = p_{ft}^* = p_{st}^*$

$$\Rightarrow \lambda = \frac{\frac{\theta}{\theta-1} p_t^{*\frac{\theta}{\theta-1}-1} \cdot p_t^*}{p_t^{*\frac{\theta}{\theta-1}}} = \frac{\theta}{\theta-1} = \lambda_1 = \lambda_2.$$

$$\Rightarrow p_t = \alpha p_{ft} + (1-\alpha) p_{st}.$$

Moreover, by labor market clearing:

$$l_t(s^t) = [\alpha (p_f(i, s^t))^{\frac{1}{\theta-1}} + (1-\alpha)(p_s(i, s^{t-1}))^{\frac{1}{\theta-1}}]\, p_t(s^t)^{\frac{1}{1-\theta}}\, y_t(s^t)$$

log linearizing this, we have

$$l^*(1+l_t) = [\alpha\, p_f^{*\frac{1}{\theta-1}}(1+\mu p_{ft}) + (1-\alpha)\, p_s^{*\frac{1}{\theta-1}}(1+\mu p_{st})]\, p_t^{*\frac{1}{1-\theta}}(1-\mu p_t)\, y^*(1+y_t)$$

In steady state, $l^* = y^*$, $p_f^* = p_t^* = p_s^*$.

$$1+l_t = (1+\mu(\alpha p_{ft} + (1-\alpha) p_{st}))(1-\mu p_t)(1+y_t)$$

$$= 1+y_t \Rightarrow l_t = y_t.$$

and $y_t(s^t) = c_t(s^t) \Rightarrow y^*(1+y_t) = c^*(c_t+1) \Rightarrow y_t = c_t = l_t$

Now, do linearization of the system.

From intertemporal Euler equation we have.

$$\beta E_t \frac{u_c(s^{t+1})}{u_c(s^t)} \frac{p_t(s^t)}{p_{t+1}(s^{t+1})} R_t(s^t) = 1$$

$$\Rightarrow \beta E_t \frac{u_c(c^*, l^*)(1+\eta_1 c_{t+1} + \eta_2 l_{t+1})}{u_c(c^*, l^*)} (1-\eta_1 c_t - \eta_2 l_t) \frac{p_t^*}{p_{t+1}^*} (1+p_t)(1-p_{t+1}) R^*(\ldots$$

$=$

and in steady state $\Rightarrow \beta \frac{p_t^*}{p_{t+1}^*} R^* = 1$

$$\Rightarrow E_t (1+(\eta_1+\eta_2)(c_{t+1}-c_t))(1+p_t-p_{t+1})(1+r_t) = 1$$

By $c_t = l_t$.

$$\Rightarrow E_t (\eta_1+\eta_2)(c_{t+1}-c_t) + p_t - p_{t+1} + r_t = 0.$$

$$\Rightarrow c_t = E_t c_{t+1} + \frac{1}{\eta_1+\eta_2}(r_t - E_t \pi_{t+1}), \quad \pi_{t+1} = p_{t+1} - p_t.$$

$$\Rightarrow c_t = E_t c_{t+1} - \left(\frac{-1}{\eta_1+\eta_2}\right)(r_t - E_t \pi_{t+1}). \quad (1).$$

define $\psi = \frac{-1}{\eta_1+\eta_2}$ and $\eta_1 = \frac{u_{cc}(c^*, l^*)\, c^*}{u_c(c^*, l^*)}$, $\eta_2 = \frac{u_{cl}(c^*, l^*)\, l^*}{u_c(c^*, l^*)}$

From cash-in-advance constraint:

$$p_t(s^t)\, c_t(s^t) = M_t(s^t).$$

$$\Rightarrow \frac{p_t^*}{p_{t+1}^*}(1+p_t-p_{t+1})(1+c_t-c_{t+1}) \frac{c^*}{c^*} = \frac{M_t^*}{M_{t+1}^*}(1+m_t-m_{t+1})$$

in steady state, $\frac{p_t^*}{p_{t+1}^*} = \frac{M_t^*}{M_{t+1}^*}$

$$\Rightarrow \mu_{t+1} \equiv m_{t+1} - m_t = c_{t+1} - c_t + p_{t+1} - p_t = \pi_{t+1} + y_{t+1} - y_t$$

$$\Rightarrow \pi_t = \mu_t - (y_t - y_{t-1}) \quad (2)$$

Flexible intermediate goods producer 's price is

$$p_{ft}(s^t) = \frac{w_t(s^t)}{\theta}. \Rightarrow p_{ft}^*(1+p_{ft}) = \frac{w_t^*}{\theta}(1+w_t)$$

$$\Rightarrow p_{ft} = w_t.$$

if we impose tax on labor, s.t. $1+\tau_t(s^t) = \frac{R_t(s^t)}{\theta}$

$$\Rightarrow -\frac{u_l(s^t)}{u_c(s^t)} = \frac{w_t(s^t)}{p_t(s^t)}\frac{1}{\theta} = \frac{p_{ft}(s^t)}{p_t(s^t)}.$$

$$\Rightarrow -\frac{u_l(c^*,l^*)(1+\eta_1 c_t+\eta_2 l_t)}{u_c(c^*,l^*)}(1-\eta_3 c_t-\eta_4 l_t)\, p_t^*(1+p_t) = p_{ft}^*(1+p_{ft})$$

where $\eta_1 = \frac{u_{lc}(c^*,l^*)c^*}{u_l(c^*,l^*)}$, $\eta_2 = \frac{u_{ll}(c^*,l^*)l^*}{u_l(c^*,l^*)}$

$\eta_3 = \frac{u_{cc}(c^*,l^*)c^*}{u_c(c^*,l^*)}$, $\eta_4 = \frac{u_{cl}(c^*,l^*)l^*}{u_c(c^*,l^*)}$

in SS, $-\frac{u_l(c^*,l^*)}{u_c(c^*,l^*)} = \frac{p_{ft}^*}{p_t^*}$

$$\Rightarrow 1+(\eta_1-\eta_3)c_t+(\eta_2-\eta_4)l_t + p_t = p_{ft}+1$$

$$\Rightarrow p_{ft} = (\eta_1+\eta_2-\eta_3-\eta_4)l_t + p_t = (\eta_1+\eta_2-\eta_3-\eta_4)y_t + p_t. \quad (3)$$

define $\eta_1+\eta_2-\eta_3-\eta_4 = \delta$.

- sticky intermediate goods producer:

$$\frac{\sum_{s^{t+1}} Q_{t+1}(s^{t+1}|s^t)\, p_{t+1}(s^{t+1})^{\frac{1}{1-\theta}}\, y_{t+1}(s^{t+1})\, w_{t+1}(s^{t+1})}{\theta \sum_{s^{t+1}} Q_{t+1}(s^{t+1}|s^t)\, p_{t+1}(s^{t+1})^{\frac{1}{1-\theta}}\, y_{t+1}(s^{t+1})} = p_s(i,s^t).$$

$$= \frac{\sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\frac{u_c(s^{t+1})}{u_c(s^t)}\, p_{t+1}(s^{t+1})^{\frac{\theta}{1-\theta}}\, y_{t+1}(s^{t+1})\, w_{t+1}(s^{t+1})}{\sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\frac{u_c(s^{t+1})}{u_c(s^t)}\, p_{t+1}(s^{t+1})^{\frac{\theta}{1-\theta}}\, y_{t+1}(s^{t+1})}\,\frac{1}{\theta}.$$

By log-linearization, we have

$$(1+p_{st})p_{st}^*(s^t) = \frac{w_{t+1}^*}{\theta}\,\frac{\sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)(1+\mu(c_{t+1}-c_t)+\frac{\theta}{\xi}p_{t+1}+y_{t+1}+w_{t+1}}{\sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)(1+\mu(c_{t+1}-c_t)+\frac{\theta}{\xi}p_{t+1}+y_{t+1})}$$

$$\Rightarrow 1 + p_{s\,t+1} = 1 + \sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\; \omega_{t+1}(s^{t+1}).$$

$$= 1 + E_t\, p_{f\,t+1} \Rightarrow p_{s\,t+1} = E_t\, p_{f\,t+1} = E_t(p_{t+1} + \delta y_{t+1})$$

this is because we can approximate:

$$\frac{\sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\,(1+\mu(c_{t+1}-c_t) + \frac{\delta}{\beta} p_{t+1} + y_{t+1} + \omega_{t+1})}{\sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\,(1+\mu(c_{t+1}-c_t) + \frac{\delta}{\beta} p_{t+1} + y_{t+1})}.$$

$$\approx \Big(1 + \sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\,(1+\mu(c_{t+1}-c_t) + \frac{\delta}{\beta} p_{t+1} + y_{t+1} + \omega_{t+1})\Big).$$

$$\Big(1 - \sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\,(1+\mu(c_{t+1}-c_t) + \frac{\delta}{\beta} p_{t+1} + y_{t+1})\Big).$$

$$\approx 1 + \sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\,\omega_{t+1}(s^{t+1}) \quad . \quad (4).$$

and (3) is already obtained when proving $c_t = l_t = y$

4. According to the description in Appendix 5.A. The economic system can be written as follows:

Firms:

Final: $\max_{y(i,s^t), P(s^t), y(i,s^{t-1})} P(s^t) y(s^t) - \int_0^\alpha P_f(i,s^t) y(i,s^t) di - \int_\alpha^1 P_s(i,s^{t-1}) y(i,s^t) di$

s.t. $y(s^t) = [\int y(i,s^t)^\theta di]^{\frac{1}{\theta}}$

Flexible: $\max_{P_f(i,s^{t-1})} [P_f(i,s^t) - W(s^t)]\, y^d(i,s^t)$

s.t. $y(i,s^t) \le l(i,s^t)$

$y^d(i,s^t) = [P(s^t)/P(i)]^{\frac{1}{1-\theta}}\, y(s^t)$

Sticky: $\max_{P_s(i,s^{t-1})} P_s(i,s^{t-1}) \sum_{s^t} Q(s^t|s^{t-1}) y^d(i,s^t) - \sum_{s^t} Q(s^t|s^{t-1}) W(s^t) y^d(i,s^t)$

s.t. $y(i,s^t) \le l(i,s^t)$

$y^d(i,s^t) = [P(s^t)/P(i)]^{\frac{1}{1-\theta}}\, y(s^t)$.

Consumer: $\max_{c(s^t), l(s^t)} \sum_{t=0}^\infty \sum_{s^t} \beta^t g(s^t)\, U(c(s^t), l(s^t))$

s.t. $P(s^t) c(s^t) = M(s^t)$.

$M(s^t) + \frac{B(s^t)}{R(s^t)} = R_p(s^{t-1})(1+\tau_l) W(s^{t-1}) l(s^{t-1})$

$+ [1-\tau(s^{t-1})] B(s^{t-1}) + T(s^t) + \Pi(s^t)$

$B(s^{t+1}) \ge \bar{B}$

From above, we can get the following:

(Flexible): $P_f(i,s^t) = \frac{1}{\theta} W(s^t)$ ①

(Sticky): $P_s(i,s^{t-1}) = \frac{1}{\theta} \frac{\sum_{s^t} Q(s^t|s^{t-1}) P(s^t)^{\frac{1}{1-\theta}} W(s^t) y(s^t)}{\sum_{s^t} Q(s^t|s^{t-1}) P(s^t)^{\frac{1}{1-\theta}}\, y(s^t)}$ ②

(Final): $P(s^t) = [\int_0^\alpha P_f(i,s^t)^{\frac{\theta}{\theta-1}} di + \int_\alpha^1 P_s(i,s^{t-1})^{\frac{\theta}{\theta-1}} di]^{\frac{\theta-1}{\theta}}$ ③

(Consumer): $\frac{1}{R_p(s^t)} = \sum_{s^{t+1}} Q(s^{t+1}|s^t)$. ④

$$Q(s^{t+1}|s^t) = \beta g(s^{t+1}|s^t) \cdot \frac{U_c(s^{t+1})\, P(s^t)}{U_c(s^t)\, P(s^{t+1})} \quad ⑤$$

$$-\frac{U_\ell(s^t)}{U_c(s^t)} = \frac{(1+\tau_\ell) W(s^t)}{P(s^t)} \quad ⑥$$

$$\frac{1}{R(s^t)} = [1-\tau(s^t)] \sum_{s^{t+1}} \beta g(s^{t+1}|s^t) \frac{U_c(s^{t+1}) P(s^t)}{U_c(s^t) P(s^{t+1})} \quad ⑦$$

$$= (1-\tau(s^t)) \frac{1}{R_p(s^t)} = (1-\tau(s^t)) \sum_{s^{t+1}} Q(s^{t+1}|s^t)$$

Therefore, competitive equilibrium is characterized by ① ~ ⑦.

More specifically:

By ⑥, ① becomes:

$$P_f(i, s^t) = \frac{-1}{\theta} \cdot \frac{U_\ell(s^t)}{U_c(s^t)} \cdot \frac{P(s^t)}{(1+\tau_\ell)} \quad ⑧$$

② becomes:

$$P_s(i, s^t) = -\frac{1}{\theta} \frac{\sum_{s^t} Q(s^t|s^{t-1}) P(s^t)^{\frac{2-\theta}{1-\theta}} \cdot \frac{U_\ell(s^t)}{U_c(s^t)} \cdot \frac{1}{1+\tau_\ell} \cdot y(s^t)}{\sum_{s^t} Q(s^t|s^{t-1}) P(s^t)^{\frac{1}{1-\theta}} y(s^t)} \quad ⑨$$

Substitute ⑧, ⑨ into ③

$$P(s^t)^{\frac{\theta}{\theta-1}} = \alpha \left(-\frac{1}{\theta} \frac{U_\ell(s^t)}{U_c(s^t)} \cdot \frac{P(s^t)}{1+\tau_\ell}\right)^{\frac{\theta}{\theta-1}}$$

$$+ (1-\alpha) \left(-\frac{1}{\theta} \frac{\sum_{s^t} Q(s^t|s^{t-1}) P(s^t)^{\frac{2-\theta}{1-\theta}} \cdot \frac{U_\ell(s^t)}{U_c(s^t)} \cdot \frac{1}{1+\tau_\ell} y(s^t)}{\sum_{s^t} Q(s^t|s^{t-1}) P(s^t)^{\frac{1}{1-\theta}} y(s^t)}\right)^{\frac{\theta}{\theta-1}} \quad ⑩$$

Then, competitive equilibrium is characterized by ⑥ ⑦ ⑧ ⑨ ⑩.

Through the parper, we know that Proposition 2 tells us that under the restricted policy of interest rate, the economy has a continuum of competitive equilibria.

Based on the same logics, we can propose a similar proposition under nonlinear economy.

$\boxed{\text{Proposition}}$: Under the restricted policy (linear feedback) of interest rate, for every feedback rule the economy has a continuum of competitive equilibria.

In order to prove it, we need to show that under the specific rule of interest rate, there exist multiple solutions satisfying 6 ~ 10. Here's how we find out the solutions:

- First, after the government observes the average sticky price, it makes the interest rate policy at period t: $i(s^{t-1})$ and it is observed by the flexible producers and the consumers: Let $R(s^t) = i(s^{t-1})$,
- then by 7, we can find the feasible set of $Q(s^{t+1}|s^t)$ s.t. $\{(Q(s^{t+1}|s^t))_{s^{t+1}\in S} \in \mathbb{R}^n \mid \sum_{s^{t+1}} Q(s^{t+1}|s^t) = \frac{1}{i(s^{t-1})(1-\tau(s^t))}\}$. This feasible set depends on the interest rate policy and the current period shocks. Also, it's not empty and not singleton.
- Meanwhile, the consumer's decision on current & future consumption can be determined by 7, depending on current shock $\tau(s^t)$, interest rate $i(s^{t-1})$ and currents price level. The current average price and flexible price and be determined by 10 and 8.

In this sense, consumption, flexible price and average price in current period can be determined via the combination of ① ~ ⑩

Also current output can be determined through the restion that $y(i, S^t) \le l(i, S^t)$, $\int_i l(i, S^t)\,di \le l(S^t)$ and,

$$y(S^t) = \left[\int y(i, S^t)^{\theta}\,di\right]^{\frac{1}{\theta}}$$

- Since these decisions are all made after shocks, then current shock $\tau(S^t)$ is embedded in all of them.

At period t+1, since $P_s(i, S^{t+1})$ depends on $Q(S^{t+1}|S^t)$ which is determined in period t. Since $Q(S^{t+1}|S^t)$ has the shock in last period, i.e. $\tau(S^t)$, also $i(S^t)$, influencing its value, then $P_s(i, S^{t+1})$ also has $\tau(S^t)$, $i(S^t)$ influencing its value.

In all, at period t sticky producers base their price decisions on $i(S^{t-1})$ and $\tau(S^{t-1})$, i.e. interest rate policy and the shock in last period. Consumer bases its decision on the average price observed and current shock $\tau(S^t)$. flexible producers based their decision on $\tau(S^t)$ and consumer's current preferences. That is, through ⑥ ~ ⑩ we can find at least one solution exist.

As we show previously, at the period t the feasible set of $Q(S^t|S^{t})$ is not a singleton. Based on this and ⑨, we can find that a group of $P_s(i, S^t)$ can be obtained accordingly. Since $P_s(i, S^t)$ influence $P(S^t)$ and thereby consumption decisions for consumers, we may find corresponding $P(S^t)$, $P_f(i, S^t)$, $C(S^t)$, $l(S^t)$ with respect to each of $P_s(i, S^t)$. Therefore, there is more than one equilibrium

HW 6 Problem 5.

(a) Define τ_1, τ_2 is the tax rate on labor income. define δ as the default coefficient on government bond at period 2. Given the tax rate, a competitive equilibrium for this economy is a consumer allocation $\overbrace{\{c_1, c_2, l_1, l_2, b^d\}}^{z^H}$ and a allocation for government $\{\bar{b}, \bar{g}\}$ and a firm $\{l_1^f, l_2^f\}$, an price system $\{\bar{q}_2, \bar{w}_1, \bar{w}_2\}$

1) Given τ_1, τ_2, δ_2 and $\{\bar{q}, \bar{w}_1, \bar{w}_2\}$, consumers choose z^H to solve

$$\max \log c_1 + \gamma \log(1-l_1) + \beta(\log c_2 + \gamma \log(1-l_2))$$

$$\text{s.t.} \quad c_1 + \bar{q} b^d \le (1-\tau_1)\bar{w}_1 l_1$$

$$c_2 \le (1-\tau_2)\bar{w}_2 l_2 + (1-\delta) b^d$$

2) Given $\{\bar{q}_2, \bar{w}_1, \bar{w}_2\}$, the firm choose $\{l_1^f, l_2^f\}$ to solve

$$\max \ \cancel{c_t} + g_t - \bar{w}_t l_t^f$$

$$\text{s.t.} \quad c_t + g_t \le A l_t^f \quad t = 1, 2.$$

which implies $\bar{w}_1 = \bar{w}_2 = A$

3) Government B.C.

$$\tau_1 \bar{w}_1 l_1 + \bar{q} b = g_1$$

$$\tau_2 \bar{w}_2 l_2 = (1-\delta) b$$

4) Market clearing:

$$c_t + g_t = A l_t \quad \forall t = 1, 2.$$

$$b^d = \hat{b}$$

$$l_t = l_t^f, \ \forall t = 1, 2$$

(b) For government's policy $\pi = (\tau_1, \tau_2, \delta)$, define F as a allocation rule, then a Ramsey equilibrium is such that

1) Given any π', the allocation rule $F(\pi')$ solves the the problem

$$\max \log c_1 + \gamma \log(1-l_1) + \beta(\log c_2 + \gamma \log(1-l_2))$$

$$\text{s.t.} \quad c_1 + \bar{q} b \le (1-\tau_1')\bar{w}_1 l_1$$

$$c_2 \le (1-\tau_2')\bar{w}_2 l_2 + (1-\delta')b$$

2) the policy π maximizes

$$\max \log c_1(\pi) + \gamma \log(1-l_1(\pi)) + \beta(\log c_2(\pi) + \gamma \log(1-l_2(\pi)))$$

$$\text{s.t.} \quad g_1 \le \tau_1 A l_1(\pi) + \bar{q} b(\pi)$$

$$(1-\delta) b(\pi) = \tau_2 A l_2(\pi)$$

(c) Assume there are a large number of small private agents.

Define history and allocation rule and strategy:

Given $h_0 = \phi$, government chooses $\tau_1 = \tau_1(\phi)$, $q = q(\phi)$

Let $h_1 = (h_0, \tau_1, q)$ agent chooses $c_1 = c_1(h_1)$, $l_1 = l_1(h_1)$, $b = b(h_1)$

Let $h_2 = (h_1, C_1, L_1, B)$, government chooses $\tau_2 = \tau_2(h_2)$, $\delta = \delta(h_2)$

Let $h_3 = (h_2, \tau_2, \delta)$, agent chooses $c_2 = c_2(h_3)$, $l_2 = l_2(h_3)$

Let $\sigma = (\tau_1(h_0), q(h_0), \tau_2(h_2), \delta(h_2))$; $f = (c_1(h_1), l_1(h_1), b(h_1),$
$c_2(h_3), l_2(h_3))$

(σ, f) is a sustainable equilibrium if

(1) • Given $\forall h_0$, $\tau_1(h_0), q(h_0), \tau_2(h_2), \delta(h_2)$ solve

(G1)

$$\max_{\substack{\tau_1, q \\ \tau_2, \delta}} \log(C_1(h_0, \tau_1, q)) + \alpha \log(1 - L_1(h_0, \tau_1, q)) + \beta[\log(C_2(h_2, \tau_2, \delta)) + \alpha \log(1 - L_2(h_2, \tau_2, \delta))].$$

$$\text{s.t.} \quad g \le \tau_1 A L_1(h_0, \tau_1, q) + q B(h_1).$$

$$(1-\delta) B(h_1) \le \tau_2 A L_2(h_2, \tau_2, \delta)$$

(G2)

• Given $\forall h_2$, $\tau_2(h_2)$, $\delta(h_2)$ solve

$$\max_{\tau_2, \delta} \log(C_1 + \alpha \log L_1) + \beta[\log C_2(h_2, \tau_2, \delta) + \alpha \log(1 - L_2(h_2, \tau_2, \delta))]$$

$$\text{s.t.} (1-\delta) B \le \tau_2 A L_2(h_2, \tau_2, \delta).$$

(P1)

(2) • Given $\forall h_1$, $c_1(h_1), l_1(h_1), b(h_1), c_2(h_3), l_2(h_3)$ solves.

$$\max_{\substack{c_1, l_1 \\ c_2, l_2}} \log(c_1 + \alpha \log(1 - l_1)) + \beta[\log(c_2) + \alpha \log(1 - l_2)]$$

$$\text{s.t.} \quad c_1 + q b \le (1 - \tau_1) A l_1 \quad (\mu_1)$$

$$c_2 \le (1 - \delta(h_1, C_1, L_1, B)) b + (1 - \tau_2(h_1, C_1, L_1, B)) A l_2$$

(μ_2).

(P2)

- Given $\forall h_3$, $c_2(h_3)$, $l_2(h_3)$ solve:

$$\max_{c_2, l_2} \log(c_1 + \alpha(1-l_1)) + \beta[\log(c_2) + \alpha\log(1-l_2)]$$

$$\text{s.t. } c_2 \le (1-\delta)B + (1-\tau_2)A\, l_2. \quad (\lambda)$$

Characterize the outcome:

First consider problem (P2).

The solution is: $\begin{cases} \frac{1}{c_2} = \lambda \\ \frac{\alpha}{1-l_2} = (1-\tau_2)A\lambda \end{cases} \Rightarrow \frac{\alpha c_2}{1-l_2} = (1-\tau_2)A.$

$\Rightarrow c_2 = \frac{(1-\tau_2)A}{\alpha}(1-l_2) \Rightarrow l_2 = \frac{1}{1+\alpha}\left[\frac{(1-\tau_2)A - (1-\delta)B\alpha}{(1-\tau_2)A}\right] = l_2(h_3)$

$\Rightarrow c_2 = \frac{(1-\tau_2)A + (1-\delta)B}{1+\alpha} = c_2(h_3).$

Then consider problem (P1).

The solution is: $\frac{\alpha c_1}{1-l_1} = (1-\tau_1)A.$; $\frac{1}{c_1} = \mu_1$, $\frac{\alpha}{1-l_1} = (1-\tau_1)A\mu_1$;

$\mu_1 q \le (1-\delta(h_1, c_1, l_1, B)))\mu_2$; $\frac{\beta}{c_2} = \mu_2$, $\frac{\alpha\beta}{1-l_2} = \mu_2(1-\tau_2(h_1, c_1, l_1 \ldots$

A

$\Rightarrow c_1 = c_1(h_1) = \frac{A(1-\tau_1) + qb(h_1)}{1+\alpha}$

$l_1 = l_1(h_1) = \frac{A(1-\tau_1) + \alpha q b(h_1)}{(1+\alpha)A(1-\tau_1)}.$

Now consider problem (G2)

The solution is:

Since $C_2(h_2, \tau_2, \delta) = c_2(h_2, \tau_2, \delta) = \dfrac{(1-\tau_2)A + (1-\delta)B}{1+\alpha}$

$L_2(h_2, \tau_2, \delta) = l_2(h_2, \tau_2, \delta) = \dfrac{1}{1+\alpha}\left[\dfrac{(1-\tau_2)A - (1-\delta)\alpha B}{(1-\tau_2)A}\right].$

$\Rightarrow \log(C_2(h_2, \tau_2, \delta)) + \alpha \log(1 - L_2(h_2, \tau_2, \delta))$

$= \log\left[\dfrac{(1-\tau_2)A + (1-\delta)B}{1+\alpha}\right] + \alpha \log\left(1 - \dfrac{1}{1+\alpha} + \dfrac{(1-\delta)B}{(1-\tau_2)A}\dfrac{\alpha}{1+\alpha}\right).$

$= \log[(1-\tau_2)A + (1-\delta)B] + \alpha \log\left(1 + \dfrac{(1-\delta)B}{(1-\tau_2)A}\right) + \alpha \log\dfrac{\alpha}{1+\alpha} - \log(1+\alpha)$

Now concentrate on government issues nonnegative debt.

$\Rightarrow (1-\delta)B = \tau_2 A L_2(h_2, \tau_2, \delta)$

$\Rightarrow$ ~~$\log(C_2(h_2, \tau_2, \delta)) + \alpha \log(1 - L_2(h_2, \tau_2, \delta))$~~

$=$ ~~$\log[(1-\tau_2)A + \tau_2 A \frac{1}{1+\alpha}(1 - (1-\delta)\alpha B$~~

$\Rightarrow L_2(h_2, \tau_2, \delta) = \dfrac{1}{1+\alpha}\left(1 - \dfrac{\alpha}{(1-\tau_2)A} \cdot \tau_2 A L_2(h_2, \tau_2, \delta)\right).$

$\Rightarrow \left[1 + \dfrac{\alpha \tau_2}{(1+\alpha)(1-\tau_2)}\right] L_2(h_2, \tau_2, \delta) = \dfrac{1}{1+\alpha}.$

$\Rightarrow L_2(h_2, \tau_2, \delta) = \dfrac{1}{(1+\alpha) + \frac{\alpha \tau_2}{1-\tau_2}} \Rightarrow (1-\delta)B = \dfrac{\tau_2 A}{(1+\alpha) + \frac{\alpha \tau_2}{1-\tau_2}}$

$\Rightarrow$ objective function $= \log\left[(1-\tau_2)A + \dfrac{\tau_2 A}{(1+\alpha) + \frac{\alpha \tau_2}{1-\tau_2}}\right]$

$+ \alpha \log\left(1 + \dfrac{\tau_2}{(1+\alpha)(1-\tau_2) + \alpha \tau_2}\right) + \alpha \log\dfrac{\alpha}{1+\alpha}$

$= \log A - \log\left(1 + \dfrac{\alpha}{1+\alpha}\dfrac{\tau_2}{1-\tau_2}\right) + \alpha \log(1+\alpha) - \alpha \log(1+\alpha-\tau_2) \quad - \log(1+\alpha)$

$+ \alpha \log\dfrac{\alpha}{1+\alpha} - \log(1+\alpha).$

$\Rightarrow$ objective function $= \log A + \log\left(1-\tau_2+\frac{\tau_2(1-\tau_2)}{1+\alpha-\tau_2}\right) + \alpha\log\frac{1+\alpha}{1+\alpha-\tau_2}$

$= \log(1-\tau_2) + \log(1+\alpha) - \log(1+\alpha-\tau_2)(1+\alpha) + k \quad + \alpha\log\frac{\alpha}{1+\alpha} - \log(1+\alpha)$

$\Rightarrow \frac{\partial of}{\partial \tau_2} = \frac{1}{1-\tau_2}(-1) + \frac{1+\alpha}{1+\alpha-\tau_2} = \frac{1}{1-\frac{\tau_2}{1+\alpha}} - \frac{1}{1-\tau_2} < 0$

$\Rightarrow \tau_2^* \equiv 0.$ and $\delta^* \equiv 1.$

Given, this we have, $B(h_1) \equiv 0. = b(h_1)$.

$\Rightarrow \tau_1^* = \frac{g}{A L_1(h_0, \tau_1, q_1)} = \frac{g}{A\frac{1}{1+\alpha}} = \frac{g(1+\alpha)}{A}$

$\Rightarrow c_1^*(h_1) = \frac{A(1-\tau_1^*)}{1+\alpha} = \frac{A\left(1-\frac{g(1+\alpha)}{A}\right)}{1+\alpha} = \frac{A-g(1+\alpha)}{1+\alpha}$

$l_1^*(h_1) = \frac{A(1-\tau_1^*)}{(1+\alpha)A(1-\tau_1^*)} = \frac{1}{1+\alpha}$

$c_2^*(h_3) = \frac{(1-\tau_2^*)A}{1+\alpha} = \frac{A}{1+\alpha}, \quad l_2^*(h_3) = \frac{1}{1+\alpha}$

$\Rightarrow$ in summary

$c_1^*(h_1) = \frac{A-g(1+\alpha)}{1+\alpha}, \quad l_1^*(h_1) = \frac{1}{1+\alpha}, \quad c_2^*(h_3) = \frac{A}{1+\alpha}, \quad l_2^*(h_3) = \frac{1}{1+\alpha}$

$\tau_1^*(h_0) = \frac{g(1+\alpha)}{A}, \quad \tau_2^*(h_1) \equiv 0, \quad \delta^*(h_1) \equiv 1.$