APEC 8212 Econometric Analysis II: Recitation Week 14

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1 Intuition of homogenous solution

We consider the ARMA(p, q) process

$$y_t = a_0 + \sum_{i=1}^{p} a_i y_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j \varepsilon_{t-j} \tag{1}$$

Homogeneous equation

The homogeneous part of (1) is

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_p y_{t-p} \tag{2}$$

then we have the following characteristic equation of "dynamic system" (2).

$$\lambda^p - a_1 \lambda^{p-1} - a_2 \lambda^{p-2} - \cdots - a_{p-1}\lambda - a_p = 0 \tag{3}$$

Suppose there are p different characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_p$, then

$$y_t^p = A_1 \lambda_1^t + A_2 \lambda_2^t + \cdots + A_p \lambda_p^t \tag{4}$$

This is the homogeneous solution of (1).

AR(p) process

Now let's look at the AR part of (1)

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_p y_{t-p} \tag{5}$$

This is also a dynamic system. Assume $a_1 + a_2 + \cdots + a_p \neq 1$, let y^* denote the steady state, we can solve that

$$y^* = \frac{a_0}{1 - a_1 - a_2 - \cdots - a_p} \tag{6}$$

Define

$$\hat{y}_t = y_t - y^*$$

then (5) can be written as the following homogeneous equation

$$\hat{y}_t = a_1 \hat{y}_{t-1} + a_2 \hat{y}_{t-2} + \cdots + a_p \hat{y}_{t-p}$$

and the solution is

$$\hat{y}_t = A_1 \lambda_1^t + A_2 \lambda_2^t + \cdots + A_p \lambda_p^t \tag{7}$$

or

$$y_t = \frac{a_0}{1 - a_1 - a_2 - \cdots - a_p} + A_1 \lambda_1^t + A_2 \lambda_2^t + \cdots + A_p \lambda_p^t$$

This is just the homogenous solution (4) plus the particular solution y^*.

To determine As, let $(y_0,\, y_1,\, \ldots,\, y_{p-1}) \in \mathbb{R}^p$ be the initial condition. Then evaluate (7) for $t = 0,\, 1,\, \ldots,\, p-1$ we have

$$\begin{pmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \vdots \\ \hat{y}_{p-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{pmatrix}$$

then

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \end{pmatrix}^{-1} \begin{pmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \vdots \\ \hat{y}_{p-1} \end{pmatrix} \tag{8}$$

Thus As are just linear combinations of $(\hat{y}_0,\, \hat{y}_1,\, \ldots,\, \hat{y}_{p-1})$, which is the deviation of initial condition $(y_0,\, y_1,\, \ldots,\, y_{p-1})$ from steady state y^*.

The stability of (5), or $\lim_{t\to 0} y_t = y^*$, is ensured when $|\lambda_i| < 1$ for any $i = 1,\, 2,\, \ldots,\, p$. Then any initial condition in $\mathbb{R}^p$ converges to the steady state. If all characteristic roots are on or outside the unit circle, (5) is unstable. When there are some λs inside the unit circle and some λs are on or outside the unit circle, (5) is usually called "saddle path stable". In other words, in the state space the set of initial conditions which converge to steady state only has measure 0.

Given the initial condition, the solution of AR(p) process (5) is equation (7), where λs are determined by the characteristic equation (3) and As are determined by (8). λs measure the speed of convergence of y_t to the steady state, and As measure the deviation of initial condition from the steady state.

2 Weakly stationary ARMA process

For ARMA(p, q) process (1) being stationary, it's required that $\mathrm{E}(y_t) = a_0 + \sum_{i=1}^{p} a_i \mathrm{E}(y_{t-i})$ is a constant over time. This implies two facts: 1) $\mathrm{E}(y_t)$ is in the steady state; and 2) y_t is stable. If $\mathrm{E}(y_t)$ is not in the steady state, then it varies over time because of the convergence to (or divergence from) the steady state. If y_t is not stable, then y_t would diverge from the steady state once it is pushed away from there by any random shock ε_t. Thus an ARMA(p, q) process is stationary only if it is in the steady state $\mathrm{E}(y_t) = y^*$. This implies either (1) has already converged to the steady state (t is sufficiently large, or the process started from infinite past), or the initial condition is just the steady state (As are all 0). Therefore, when we discuss the stationarity of any ARMA process, we can always ignore the homogenous solution because it is 0.

Then we can define the stationarity as the following

$$\begin{aligned} \mathrm{E}(y_t) &= y^* \qquad \forall t \\ \mathrm{Var}(y_t) &= \mathrm{E}(y_t - y^*)^2 = \sigma^2 \qquad \forall t \qquad (9) \\ \mathrm{Cov}(y_t,\, y_{t-s}) &= \mathrm{E}(y_t - y^*)(y_{t-s} - y^*) = \gamma_s \qquad \forall t,\, s \qquad (10) \end{aligned}$$

Let's write (1) in the compact form

$$A_p(L) y_t = a_0 + B_q(L)\varepsilon_t$$

where

$$\begin{aligned} A_p(L) &= 1 - a_1L - a_2L^2 - \cdots - a_pL^p \\ B_q(L) &= 1 + b_1L + b_2L^2 + \cdots + b_qL^q \end{aligned} \tag{11}$$

Comparing (11) with characteristic equation (3) we find that (3) is

$$A_p(1/\lambda) = 0$$

Then (1) is stable if the roots of $A_p(L)$ are all outside the unit circle.

According (6)

$$y^* = a_0/A_p(1)$$

so

$$y_t = y^* + \frac{B_q(L)}{A_p(L)}\varepsilon_t = y^* + x_t$$

where

$$\begin{aligned} x_t &= c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + c_3\varepsilon_{t-3} + \cdots \\ &= \varepsilon_t(c_0 + c_1L + c_2L^2 + c_3L^3 + \cdots) \end{aligned}$$

is a MA(∞) process. Also we know

$$\mathrm{Var}(y_t) = \mathrm{Var}(x_t) \qquad \mathrm{Cov}(y_t,\, y_{t-s}) = \mathrm{Cov}(x_t,\, x_{t-s})$$

For (9) and (10) being well-defined we need

$$\begin{aligned} M &= c_1^2 + c_2^2 + c_3^2 + \cdots \\ N &= c_s + c_{s+1}c_1 + c_{s+2}c_2 + \cdots \end{aligned}$$

to be finite (notice that this is always true for any MA(q) process where $q \in \mathbb{N}$). To solve cs, use the method of undetermined coefficients and the following equation

$$B_q(L) = A_p(L)x_t$$

so

$$1 + b_1L + b_2L^2 + \cdots + b_qL^q = (1 - a_1L - a_2L^2 - \cdots - a_pL^p)(c_0 + c_1L + c_2L^2 + c_3L^3 + \cdots)$$

Expand the right hand side, for example, suppose $p > q$, we have

$$\begin{array}{ccccccccccccccc} & 1 & + & b_1L & + & b_2L^2 & +\cdots+ & b_qL^q & & & & & \\ = & c_0 & + & c_1L & + & c_2L^2 & +\cdots+ & c_qL^q & +\cdots+ & c_{p-1}L^{p-1} & + & c_pL^p & +\cdots \\ & & - & a_1c_0L & - & a_1c_1L^2 & -\cdots- & a_1c_{q-1}L^q & -\cdots- & a_1c_{p-2}L^{p-1} & - & a_1c_{p-1}L^p & -\cdots \\ & & & & - & a_2c_0L^2 & -\cdots- & a_2c_{q-2}L^q & -\cdots- & a_2c_{p-3}L^{p-1} & - & a_2c_{p-2}L^p & -\cdots \\ & & & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & - & a_qc_0L^q & -\cdots- & a_qc_{p-q-1}L^{p-1} & - & a_qc_{p-q}L^p & -\cdots \\ & & & & & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & - & a_{p-1}c_0L^{p-1} & - & a_{p-1}c_1L^p & -\cdots \\ & & & & & & & & & & - & a_pc_0L^p & -\cdots \end{array}$$

From last equation we see that, in general, for any $i > \max\{p-1,\, q\}$, the coefficients on L^i satisfies the following

$$c_i = a_1c_{i-1} + a_2c_{i-2} + \cdots + a_pc_{i-p}$$

Therefore, when i is sufficiently large, the sequence of coefficients c_i follows an AR(p) process which is exactly the same as the homogenous part of (1), thus

$$c_i = A_1\lambda_1^i + A_2\lambda_2^i + \cdots + A_p\lambda_p^i \tag{12}$$

where λs are determined by the characteristic equation (3), and As are determined by the following equation

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \end{pmatrix}^{-1} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{pmatrix}$$

Then if (1) is stable, all λs are in the unit circle, thus by (12) M and N are finite. (Notice that we assume that all characteristic roots λs are different. If there are repeated characteristic roots, the proof would be a little bit more involved.)

Therefore, for any p, $q \in \mathbb{N}$, ARMA(p, q) is weakly stationary if: 1) it's stable; and 2) it has converged to or started from the steady state.